JOURNAL OF FUNCTIONAL ANALYSIS 33, 36–46 (1979)

Existence of Nonzero Fixed Points for Noncompact Mappings in Wedges and Cones

W. V. PETRYSHYN*

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

Communicated by A. P. Calderón

Received December 10, 1977

INTRODUCTION

The purpose of this paper is to establish the existence of nonzero fixed points for mappings which are either multivalued and condensing on subsets of wedges of real Banach spaces (see Section 1) or are k-set-contractive and differentiable either at 0 or at ∞ along a given cone (see Section 2). As will be seen below, the boundary conditions under which the existence of such fixed points is established are so general that they include most if not all of existence results of this type obtained by other authors for special classes of maps and under more restrictive boundary conditions.

To be more specific, let X be a real Banach space, $K \subseteq X$ a wedge, D^1 and D^2 bounded neighborhoods of $0 \in X$ with $\bar{D}^1 \subset D^2$, $D_K{}^i = D^i \cap K$ for $i = 1, 2$, and $T: \bar{D}_K{}^2 \to 2^K$ an upper semicontinuous, multivalued, condensing mapping. Using an improved version of Theorem 1 in [7] and the index theory for multivalued condensing mappings developed by Fitzpatrick and Petryshyn in [6], it is shown in Theorem 1 that if there exist a condensing map $C: \bar{D}_K{}^1 \to 2^K$ and a compact map $F: \bar{D}_K{}^2 \to 2^K$ such that $\mu x \notin C(x)$ for $\mu > 1$ with $x \in \partial_K D^1$ and $\| w \| \geqslant \alpha > 0$ for $w \in F(x)$ with $x \in \partial_K D^2$, then under certain additional conditions on T, K, C and F there exists $x_0 \in \bar{D}_K{}^2 \backslash \bar{D}_K{}^1$ such that $x_0 \in T(x_0)$. Here $\partial_K D^i$ denotes the boundary of D_K related to K. It is shown in Section 1 that Theorem 1 extends, on the one hand, the corresponding results of Krasnoselskii [11], Turner [18], Gustafson–Schmitt [10] and Gatica–Smith [9] for single-valued compact maps defined on cones $K \subset X$ and, on the other, the results of Edmunds–Potter–Stuart [5], Nussbaum [14] and Potter [16] for singlevalued condensing and k-set-contractive maps as well as the results for multivalued condensing maps of Fitzpatrick–Petryshyn [6] and Milojevič [12].

In Section 2 we use Theorem 1 to establish the existence of nonzero fixed

* Supported in part by the NSF Grant MCS 76 06352, A01.

0022-1236/79/070036-11$02.00/0

points for k-set-contractive maps $T: K \to K$ with $k < 1$, where T is assumed to have either the Frechet derivative T_0 at 0 along K or the asymptotic derivative T_∞ at ∞ along K or both.

Indeed, Corollaries 2 and 3 to Theorem 1 provide proper extensions of the corresponding recent results of Amann [1] and of Gatica–Smith [9] as well as of the earlier results of Krasnoselskii [11], Edmunds–Potter–Stuart [5] and Amann [2]. For the detailed description of our extensions see Section 2.

1

Let X be a real Banach space. If $Q \subset X$ is any bounded set we define the *set-measure of noncompactness* of Q, $\gamma(Q)$, to be $\inf\{d > 0 \mid Q$ can be covered by a finite number of sets, each of which has diameter less than $d\}$. Clearly, $\gamma(Q) = 0$ if and only if Q is compact. It was shown by Darbo [4] that if Q_1 and Q_2 are bounded, then $\gamma(Q_1 \cup Q_2) = \max\{\gamma(Q_1), \gamma(Q_2)\}$, $\gamma(Q_1 + Q_2) \leqslant \gamma(Q_1) + \gamma(Q_2)$ $\gamma(Q_1) \leqslant \gamma(Q_2)$ if $Q_1 \subseteq Q$, $\gamma(\lambda Q) = |\lambda| \gamma(Q)$ for $\lambda \in R$ and $\gamma(\overline{\text{co}}(Q)) = \gamma(Q)$, where $\overline{\text{co}}(Q)$ is the closure of the convex hull of Q. In analogy with [8, 17] (resp. [4]) we say that an upper semicontinuous (u.s.c.) mapping $T: D \subseteq X \to 2^X$ such that $T(x)$ is closed and convex is called *condensing* (resp. *k-set-contractive*) provided that $\gamma(T(Q)) < (Q)$ for each $Q \subset D$ with $\gamma(Q) \neq 0$ (resp. $\gamma(T(Q)) \leqslant k\gamma(Q)$ for each $Q \subset D$ and some $k \geqslant 0$). If K, $D \subseteq X$ we denote the boundary and the closure of $D_K = D \cap K$ relative to K by $\partial_K D$ and $\bar{D}_K$ respectively.

If $K \subseteq X$ is closed and convex, $D \subseteq X$ is open and $T: \bar{D}_K \to 2^K$ is condensing and such that $x \notin T(x)$ if $x \in \partial_K D$, then it has been shown by Fitzpatrick and Petryshyn [6] that there exists an integer $i_K(T, D_K)$, the *fixed point index of T on D_K*, which has the following properties:

(P1) If $i_K(T, D_K) \neq 0$, then T has a fixed point in D_K.

(P2) If $x_0 \in D_K$, then $i_K(\hat{x}_0, D_K) = 1$, where $\hat{x}_0$ denotes the mapping whose constant value is x_0.

(P3) If $D = D_1 \cup D_2$, where D_1 and D_2 are disjoint open subsets such that $x \notin T(x)$ if $x \in \partial_K D_1 \cup \partial_K D_2$, then $i_K(T, D_K) = i_K(T, D_{1K}) + i_K(T, D_{2K})$.

(P4) If $H: [0, 1] \times \bar{D}_K \to 2^K$ is u.s.c. and such that $\gamma(H([0, 1] \times Q)) < \gamma(Q)$ for $Q \subset \bar{D}_K$ with $\gamma(Q) \neq 0$ and if $x \notin H(t, x)$ for $t \in [0, 1]$ and $x \in \partial_K D$, then $i_K(H(1, \cdot), D_K) = i_K(H(0, \cdot), D_K)$.

These properties were established in [6] in the more general setting when X is a Frechét space. The index defined above is an extension of the indices developed by Sadovskii [17] and Nussbaum [13] in the above situation where the mappings involved are single-valued. The properties (P1)–(P4) and an improved version of Theorem 1 from [7] will play an essential role in the proof of our main result, Theorem 1 below, which establishes the existence of nonzero fixed points for multivalued condensing mappings acting on subsets of wedges of X and satisfying very general boundary conditions.

We recall that $K \subseteq X$ is called a *wedge* provided that K is closed and such that $\alpha x + \beta y \in K$ whenever $\alpha, \beta \in [0, \infty)$ and $x, y \in K$. A wedge is called a *cone* if $K \cap (-K) = \{0\}$. A cone is called *total* if $X = \overline{K - K}$ and *generating* or *reproducing* if $X = K - K$. If K is a cone, we write $x \leqslant y$ iff $y - x \in K$. The norm $\|\cdot\|$ of X is said to be *monotone* with respect to K if for $x, y \in K$ with $x \leqslant y$ we have $\| x \| \leqslant \| y \|$. An u.s.c. mapping $T: D \subset X \to 2^X$ such that $T(x)$ is closed and convex is called *compact* provided that $T(Q)$ is relatively compact for each bounded $Q \subset D$. It is obvious that a compact map is condensing. For other examples of condensing maps see [3, 6, 8, 13, 17].

To prove our main result we need the following proposition which is a slight extension of Theorem 1 in [7].

PROPOSITION 1. *Suppose that the wedge $K \subseteq X$ is not a finite dimensional subspace of X and $D \subset X$ is a bounded neighborhood of $0 \in X$. Let $T: \bar{D}_K \to 2^K$ be condensing and $F: \bar{D}_K \to 2^K$ compact. Suppose there is $\alpha > 0$ such that $\| y \| \geqslant \alpha$ for $y \in F(x)$ with $x \in \partial_K D$ and $i_K(T, D_K) \neq 0$. Then there is a $\lambda > 0$ and $x \in \partial_K D$ such that $x \in T(x) + \lambda F(x)$.*

Proposition 1 has been proved in [7, Theorem 1] under the assumptions that $T: \bar{D}_K \to 2^K$ is k-set-contractive with $k \in [0, 1)$ and that K is either a cone or $K \cap \bar{B}(0, 1)$ is noncompact. E. N. Dancer called the writer's attention to the fact that Theorem 1 in [7] remains valid without any change in its proof if the condition on K in the last sentence of that theorem is replaced by the slightly weaker assumption (used in Proposition 1) that the wedge K is not a finite dimensional subspace of X. Moreover, a closer examination of the proof of Theorem 1 in [7] reveals that the same arguments apply when T is assumed to be only condensing (as in Proposition 1) and consequently we omit its proof.

THEOREM 1. *Let K be a wedge in X such that K is not a finite dimensional subspace of X, D^1 and D^2 bounded neighborhoods of 0 with $\bar{D}^1 \subset D^2$ and $T: \bar{D}_K^2 \to 2^K$ condensing. Suppose also that the following conditions hold:*

(i) *There is a condensing map $C: \bar{D}_K^1 \to 2^K$ such that $\mu x \notin C(x)$ if $x \in \partial_K D^1$ and $\mu > 1$ (or $\| y \| \leqslant \| x \|$ for $y \in C(x)$ with $x \in \partial_K D^1$) and $x \notin \lambda T(x) + (1 - \lambda)C(x)$ if $x \in \partial_K D^1$ and $\lambda \in [0, 1)$.*

(ii) *There is a compact map $F: \bar{D}_K^2 \to 2^K$ and $\alpha > 0$ such that $\| y \| \geqslant \alpha$ for $y \in F(x)$ with $x \in \partial_K D^2$ and $x \notin T(x) + \mu F(x)$ if $x \in \partial_K D^2$ and $\mu > 0$.*

(A1) *Then there exists $x_0 \in \bar{D}_K^2 \setminus \bar{D}_K^1$ such that $x_0 \in T(x_0)$.*

(A2) *The same assertion is true if we assume that condition* (i) *holds on D_K^2 while* (ii) *holds on D_K^1.*

Proof. (A1). Let us define $H: [0, 1] \times \bar{D}_K{}^1 \to 2^K$ by $H(t, x) = tC(x)$ for $t \in [0, 1]$ *and* $x \in \bar{D}_K{}^1$. Since, for each $Q \subset \bar{D}_K{}^1$, $H([0, 1] \times Q) \subseteq \overline{\text{co}}(C(Q) \cup \{0\})$ and C is condensing we have $\gamma(H([0, 1] \times Q)) \leqslant \gamma(C(Q)) < \gamma(Q)$ if $\gamma(Q) \neq 0$. Further, condition (i) implies that $x \notin H(t, x)$ for $t \in [0, 1]$ and $x \in \partial_K D^1$. Indeed, if not, then there would exist $t_0 \in [0, 1]$ and $x_0 \in \partial_K D^1$ such that $x_0 \in H(t_0, x_0)$, i.e., $x_0 = t_0 y$ for some $y \in C(x_0)$. Since $0 \in D^1$ and $x_0 \in \partial_K D^1$ it follows that $t_0 \neq 0$. The last assumption in (i) implies that $t_0 \neq 1$. Hence $\| x_0 \| = t_0 \| y \|$ with $x_0 \in \partial_K D^1$ and $t_0 \in (0, 1)$ which is impossible. This shows that $x \notin H(t, x)$ for $t \in [0, 1]$ and $x \in \partial_K D^1$. Thus, by Properties (4) and (2), $i_K(C, D_K{}^1) = i_K(\hat{0}, D_K{}^1) = 1$.

Consider now the map $G: [0, 1] \times \bar{D}_K{}^1 \to 2^K$ given by $G(t, x) = tT(x) + (1 - t)C(x)$. Since, for each $Q \subset \bar{D}_K{}^1$, $G([0, 1] \times Q) \subseteq \overline{\text{co}}(T(Q) \cup C(Q))$, it follows that $\gamma(G([0, 1] \times Q)) \leqslant \max\{\gamma(T(Q)), \gamma(C(Q))\} < \gamma(Q)$, if $\gamma(Q) \neq 0$. Now we may assume without loss of generality that $x \notin G(1, x)$ if $x \in \partial_K D^1$. This and the second condition in (i) imply that $x \notin G(t, x)$ for $t \in [0, 1]$ and $x \in \partial_K D^1$. Hence, by Property (4), $i_K(T, D_K{}^1) = i_K(C, D_K{}^1)$ and therefore $i_K(T, D_K{}^1) = 1$. On the other hand, in view of condition (ii) and the nonrestrictive assumption that $x \notin T(x)$ for $x \in \partial_K D^2$, Proposition 1 with $D_K = D_K{}^2$ implies that $i_K(T, D_K{}^2) = 0$. Indeed, if $i_K(T, D_K{}^2)$ were not equal to 0, then the compact map $F: \bar{D}_K{}^2 \to 2^K$, K and T satisfy all the conditions of Proposition 1 and consequently there would exist $\mu_o > 0$ and $x_0 \in \partial_K D^2$ such that $x_0 \in T(x_0) + \mu_o F(x_0)$, in contradiction to the last assumption in (ii).

Now, let $G = D^2 \backslash D^1$ and observe that $\partial_K G = \partial_K D^1 \cup \partial_K D^2$ and T has no fixed points on $\partial_K G$. Thus, by Property (3), $i_K(T, D_K{}^2) = i_K(T, D_K{}^1) + i_K(T, G_K)$. It follows from this equality that $i_K(T, G_K) = -1$. Hence, by Property (1), there exists $x_0 \in \bar{G}_K = \bar{D}_K{}^2 \backslash \bar{D}_K{}^1$ such that $x_0 \in T(x_0)$.

(A2) To prove the second part of Theorem 1 note that if condition (i) holds on $D_K{}^2$, then by the same arguments as above we see that $i_K(T, D_K{}^2) = 1$. Similarly, if (ii) holds on $D_K{}^1$, then in view of Proposition 1 with $D_K = D_K{}^1$ we see that $i_K(T, D_K{}^1) = 0$. The Property (3) implies then in this case that $i_K(T, G_K) = 1$ and so again, as in the preceding case, there exists $x_0 \in \bar{G}_K$ such that $x_0 \in T(x_0)$. Q.E.D.

An immediate consequence of Theorem 1 is the following practically useful corollary, which unifies and extends the corresponding results obtained independently (and by different arguments) by Turner [18, Theorems 3.12 and 3.15] for T, C, F single-valued and compact and by Fitzpatrick and Petryshyn [6, Corollary 3.3] for T multivalued and condensing, $C = 0$ and $F(x) = w \in K \backslash \{0\}$.

COROLLARY 1. *Let $K \subseteq X$ be as in Theorem 1, $r_1, r_2 \in (0, \infty)$ with $r = \max\{r_1, r_2\}$ and $T: \bar{B}_K(0, r) \to 2^K$ condensing. Suppose that:*

(i) *There is a condensing map $C: \bar{B}_K(0, r_1) \to 2^K$ such that $\mu x \notin C(x)$ if*

$x \in \partial_K B(0, r_1)$ and $\mu > 1$ (or $\|y\| \leqslant r_1$ if $y \in C(x)$ and $x \in \partial_K B(0, r_1)$) and $x \notin \lambda T(x) + (1 - \lambda)C(x)$ if $x \in \partial_K B(0, r_1)$ and $\lambda \in [0, 1)$.

(ii) *There is a compact map F: $\bar{B}_K(0, r_2) \to 2^K$ and $\alpha > 0$ such that $\|y\| \geqslant \alpha$ for $y \in F(x)$ with $x \in \partial_K B(0, r_2)$ and $x \notin T(x) + \mu F(x)$ if $x \in \partial_K B(0, r_2)$ and $\mu > 0$.*

Then T has a fixed point $x_0 \in K$ with $\min\{r_1, r_2\} \leqslant \|x_0\| \leqslant \max\{r_1, r_2\}$.

Let us add in passing that Theorem 1 and its corollary remain valid if instead of set-measure of noncompactness $\gamma(Q)$ we use the *ball-measure of noncompactness* $\chi(Q)$ defined as follows: $\chi(Q) = \inf\{r > 0 \mid Q$ can be covered by a finite number of balls with radii $\leqslant r\}$. This is the case because Proposition 1 and the index theory developed in [6] are also valid for condensing maps defined by χ.

Remark 1. Theorem 1 and Corollary 1 are new results even when T and C are singlevalued or when T and C are compact. The cases when $K = X$ or K is a cone are most useful in applications.

Remark 2. It was mentioned in the Introduction that Theorem 1 extends and unifies a number of theorems concerning the existence of nonzero fixed points obtained earlier by other authors, very often using different arguments. We shall indicate now some of these special cases.

(a) Suppose first that T and C are compact. Then Corollary 1 includes [18, Theorems 3.12 and 3.15] where it was additionally assumed that K is a reproducing cone and all mappings are singlevalued. If in Corollary 1 we let T be single-valued, K a cone and assume that (i) holds with $C = 0$ and (ii) holds for $F = T$, then we deduce [9, Theorems 1.2 and 1.3] (attributed in [9] to Gustafson and Schmitt) which are extensions of [18, Corollaries 10 and 11]. Furthermore, Corollary 1 also extends [9, Theorems 1.4 and 1.5] where it is assumed that T: $K \to K$ is single-valued and that either (i) holds on $B_K(0, r_1)$ with $C = 0$ and (ii) holds on $B_K(0, r_2)$ with $r_1 > r_2$ or that (i) holds on $B_K(0, r_2)$ and (ii) holds on $B_K(0, r_1)$ with $r_2 > r_1$. In particular, if we choose $k_0 \in K \backslash \{0\}$ and define $F(x) \equiv k_0$ for $x \in B_K(0, r_1)$ for the case when $r_2 > r_1$ and set $C = 0$ on $B_K(0, r_2)$, then we obtain [10, Theorem 2.6]. We add that [10, Theorem 2.5] is also a special case of Corollary 1 if we take $k, h \in K$ with $\|k\| < r_1$ and $\|h\| > r_2 > r_1$ and set $C(x) \equiv k$ for $x \in B_K(0, r_1)$ and $F(x) \equiv h$ for $x \in B_K(0, r_2)$ and observe that the monotonity of the norm with respect to the cone K assumed in [10] implies that condition (2.2) of Theorem 2.5 is valid for all $\lambda > 0$. Finally, it is easy to see that [11, Theorems 4.12 and 4.14] also follow as special cases of Corollary 1. This fact was already noted in [6, 13, 18].

(b) We now suppose that T and C are condensing. In this case Corollary 1 includes [6, Corollary 3.3] if we choose $C(x) \equiv 0$ for $x \in \bar{B}_K(0, r_1)$ in (i) and $F(x) \equiv w \in K \backslash \{0\}$ for $x \in \bar{B}_K(0, r_2)$ in (ii). In case T is single-valued the preceding result (as was noted in [6]) yields [14, Lemma 3.3]. It should be noted that an earlier result of [5, Theorem 5] for single-valued k-set-contractions with $k < 1$

is also included in Corollary 1 (see [6]). It should be added, however, that the authors of [5] had to assume that T maps all of X into K and their boundary conditions are more stringent than those of [14]. The same can be said about the result of [16]. Finally, Theorem 1 extends a recent result of [12, Theorem 3.6] where it is assumed that $C = 0$ on D_K^1, $F(x) \equiv w$ for $x \in \bar{D}_K^2$ and some $w_0 \in K \setminus \{0\}$, and the index theory of [6] is used.

Remark 3. It should be added that the proof of Theorem 1 shows that $i_K(T, G_K)$ is either 1 or -1 (where G_K is a conical shell if $D^1 = B(0, r_1)$ and $D^2 = B(0, r_2)$). This additional information can be very useful in applications since in some cases it may help us to determine whether T has a second fixed point in G_K under the assumption that T has no fixed point on $\partial_K D^1$ and on $\partial_K D^2$.

2

In order to deduce from Theorem 1 the existence of nonzero fixed points for k-set-contractive mappings which are differentiable either at 0 or at ∞ along a given cone we first recall some definitions and necessary known results.

Let K be a total cone. A map $T: K \to X$ is said to be *differentiable at* 0 *along* K if there is $T_0 \in L(X, X)$ such that for each h in K

$$T(h) = T(0) + T_0 h + w(0, h) \quad \text{with} \quad w(0, h) = o(\| h \|) \quad \text{as} \quad \| h \| \to 0. \tag{2.1}$$

The map T is said to be *asymptotically linear along* K if there is $T_\infty \in L(X, X)$ such that for each h in K

$$T(h) = T_\infty h + w(h) \quad \text{with} \quad w(h) = o(\| h \|) \quad \text{as} \quad \| h \| \to \infty. \tag{2.2}$$

It is not hard to show that since T is a total cone the maps T_0 and T_∞, *the derivatives at* 0 *and at* ∞ *along* K respectively, are uniquely determined. It was noted in [2] that the assumption that K is total, which guarantees the uniqueness of the derivative along K, contains no loss of generality since, in a more general setting, we may always restrict our consideration to the closed subspace $\overline{K - K}$ of X. It was shown by Amann [2] that if $T: K \to X$ is k-set-contractive and (2.2) holds, then $T_\infty|_K$ is also k-set-contractive. As was noted in [2], the argument of Nussbaum implies the same for $T_0|_K$ if (2.1) holds.

In what follows we say that $A \in L(X, X)$ is of *type* $L_1^+(X)$ if A has an eigenvector $h_0 \in K \setminus \{0\}$ belonging to some sigenvalue $\lambda_0 > 1$ and A has no eigenvectors in $K \setminus \{0\}$ with eigenvalues equal to 1 (see [2]).

Before we apply Theorem 1 to obtain the existence of nonzero fixed points for k-set-contractive maps satisfying either (2.1) or (2.2) we will need the following simple fact (see [11]). If $u, v \in K$, $u \neq 0$ and if $\{\lambda_n\} \subset (0, \infty)$ is a sequence such that $\lambda_n \to \infty$, then there exists $n_0 \in N$ such that $v - \lambda_{n_0} u \notin K$. Indeed, if we suppose that $v - \lambda_n u \in K$ for all n, then $(1/\lambda_n)v - u \in K$ for all n and this implies that $-u \in K$, which is impossible.

COROLLARY 2. *Let K be a total cone in X and let $T: K \to K$ be k-set-contractive with $k \in [0, 1)$ such that*

(j) *There is an $r > 0$ and a k_1-set-contraction $C: \bar{B}_K(0, r) \to K$ with $k_1 < 1$ such that $Cx \neq \mu x$ if $x \in \partial_K B(0, r)$ and $\mu > 1$ (or $\| Cx \| \leqslant r$ for $x \in \partial_K B(0, r)$) and $x \neq \lambda T(x) + (1 - \lambda)C(x)$ if $x \in \partial_K B(0, r)$ and $\lambda \in [0, 1)$.*

Suppose further that either (jj_0) *or* (jj_∞) *holds, where*

(jj_0) *$T(0) = 0$, T has the derivative T_0 at 0 along K, and T_0 is of type $L_1^+(X)$.*

(jj_∞) *T has the derivative T_∞ at ∞ along K and T_∞ is of type $L_1^+(X)$. Then, in either case, T has a fixed point in $K \setminus \{0\}$.*

Proof. First note that (j) implies condition (i) of Corollary 1 with $D^1 = B(0, r)$. Thus, to deduce Corollary 2 from Corollary 1, it suffices to show that condition (ii) of Corollary 1 is implied by either (jj_0) or (jj_∞).

Since the arguments are essentially identical in both cases, we prove these implications simultaneously for T satisfying either (jj_0) or (jj_∞). To clarify the notation, in what follows it is always assumed that either $\alpha = 0$ or $\alpha = \infty$, i.e., $\alpha = 0$ corresponds to condition (jj_0) while $\alpha = \infty$ corresponds to condition (jj_∞).

First note that (jj_0) and (jj_∞) can be put in the form

(jj_α) $T(h) = T_\alpha(h) + Q_\alpha(h)$ with $Q_\alpha(h) = o(\| h \|)$ as $\| h \| \to \alpha$ $(h \in K)$,

where $T_\alpha \in L(X, X)$ is such that $T_{\alpha|K}$ is k-set-contractive and thus $(I - T_\alpha)_{|K}$ is proper (see [13]) and α is either 0 or ∞.

We assert that we can choose an $r_\alpha > 0$ with $r_0 < r$ and $r_\infty > r$ and define the compact map $F_\alpha: \bar{B}_K(0, r_\alpha) \to K$ by $F_\alpha(x) \equiv k_\alpha$ for $x \in \bar{B}_K(0, r_\alpha)$ such that obviously $\| F_\alpha x \| = \| k_\alpha \| > 0$ for $x \in \partial_K B(0, r_\alpha)$ and $x \neq T(x) + \mu F(x)$ if $x \in \partial_K B(0, r_\alpha)$ and $\mu > 0$, where $k_\alpha \in K \setminus \{0\}$ is an eigenvector of T_α corresponding to some eigenvalue $\lambda_\alpha > 1$ of T_α and α is either 0 or ∞. The latter exists since T_α is of type $L_1^+(X)$.

If our assertion were not ture, then there would exist sequences $\{r_n^\alpha\}$ and $\{\mu_n^\alpha\}$ in $(0, \infty)$ and $\{y_n^\alpha\} \subset K$ such that $r_n^\alpha \to \alpha$ as $n \to \infty$ with $\alpha = 0$ or $\alpha = \infty$, $\mu_n^\alpha > 0$ and $\| y_n^\alpha \| = r_n^\alpha$ for each $n \in N$, and $y_n^\alpha = T(y_n^\alpha) + \mu_n^\alpha k_\alpha$. It follows from the last equality and (jj_α) that

$$y_n^\alpha = T_\alpha(y_n^\alpha) + Q_\alpha(y_n^\alpha) + \mu_n^\alpha k_\alpha \qquad \text{for each} \quad n \in N. \tag{2.3}$$

Dividing (2.3) by $\| y_n^\alpha \| = r_n^\alpha$ and setting $z_n^\alpha = y_n^\alpha / \| y_n^\alpha \|$ we get

$$(I - T_\alpha)(z_n^\alpha) = \frac{Q_\alpha(y_n^\alpha)}{\| y_n^\alpha \|} + \frac{\mu_n^\alpha}{r_n^\alpha} k_\alpha \qquad \text{for each} \quad n \in N. \tag{2.4}$$

Since $\| z_n^\alpha \| = 1$ and $\| Q_\alpha(y_n^\alpha)\|/\| y_n^\alpha \| \to 0$ as $\| y_n^\alpha \| \to \alpha$, it follows from (2.4) that $\{\mu_n^\alpha/r_n^\alpha\}$ is bounded. Hence, without loss of generality, we may assume that $\mu_n^\alpha/r_n^\alpha \to \eta^\alpha \geqslant 0$ as $n \to \infty$. Consequently, it follows from this and (2.4) that $(I - T_\alpha)(z_n^\alpha) \to \eta^\alpha k_\alpha$ as $n \to \infty$. Since $(I - T_\alpha)_{|K}$ is proper we may assume that $z_n^\alpha \to z^\alpha$ in K, $\| z^\alpha \| = 1$, and

$$(I - T_\alpha)\, z^\alpha = \eta^\alpha k_\alpha \text{ with } \eta^\alpha > 0 \qquad \text{for} \quad \alpha = 0 \text{ or } \alpha = \infty. \tag{2.5}$$

It follows from (2.5) that $z^\alpha - \eta^\alpha k_\alpha = T_\alpha(z^\alpha) \in K$ since it is known (see [11]) that $T_\alpha(K) \subset K$.

Now since $k_\alpha \in K\backslash\{0\}$ is an eigenvector of T_α corresponding to the eigenvalue $\lambda_\alpha > 1$, it follows from (2.5) or, equivalently, from the equality $z^\alpha - \eta^\alpha k_\alpha = T_\alpha(z^\alpha)$ that $T_\alpha(z^\alpha) - \eta^\alpha T_\alpha(k_\alpha) = T_\alpha(T_\alpha z^\alpha) \equiv u_\alpha \in K$, i.e., $z^\alpha - \eta^\alpha(1 + \lambda_\alpha)\, k_\alpha = u_\alpha \in K$. It follows from this and the induction argument that

$$z^\alpha - \eta^\alpha(1 + \lambda_\alpha + \lambda_\alpha^2 + \cdots + \lambda_\alpha^n)\, k_\alpha \in K \qquad \text{for each} \quad n \in N. \tag{2.6}$$

with α either 0 or ∞. Since $\eta^\alpha > 0$ and $\lambda_\alpha > 1$ the relation (2.6) is impossible by the remark preceding Corollary 2. Thus if (jj$_0$) holds, then (ii) of Corollary 1 is satisfied with $D^2 = B(0, r_0)$ and $F_0(x) \equiv k_0$ for $x \in \bar{B}_K(0, r_0)$. On the other hand, if (jj$_\infty$) holds, then (ii) is satisfied with $D^2 = B(0, r_\infty)$ and $F_\infty(x) \equiv k_\infty$ for $\bar{B}_K(0, r_\infty)$. Hence Corollary 2 follows from Corollary 1. Q.E.D.

To state our next result we first define $A \in L(X, X)$ to be of type $L_1^-(X)$ if A has no eigenfunctions in $K\backslash\{0\}$ belonging to eigenvalues $\lambda \geqslant 1$ (see [2]).

Corollary 3. *Let K be a total cone in X and let $T\colon K \to K$ be k-set-contractive with $k \in [0, 1)$ such that*

(*ll*) *There exists $r > 0$, a compact map $F\colon \bar{B}_K(0, r) \to K$ and $\alpha > 0$ such that $\| Fx \| \geqslant \alpha$ for $x \in \partial_K B(0, r)$ and $x \neq Tx + \mu Fx$ if $x \in \partial_K B(0, r)$ and $\mu > 0$.*

Then T has a fixed point in $K\backslash\{0\}$ provided that one of the following two conditions holds:

(l_0) *$T(0) = 0$, T has the derivative T_0 at 0 along K and T_0 is of type $L_1^-(X)$.*

(l_∞) *T has the derivative R_∞ at ∞ along K and T_∞ is of type $L_1^-(X)$.*

Proof. To deduce Corollary 3 from Corollary 1, it suffices to show that (i) of Corollary 1 is implied by either (l_0) or (l_∞) since (*ll*) implies (ii) with $D^2 = B(0, r)$.

In fact, we will show that there exists $r_\alpha > 0$ with $r_0 < r$ and $r_\infty > r$ such that (i) holds on any $B(0, \sigma) \subseteq B(0, r_0)$ with $C = T_0$ if (l_0) holds, while (i) holds on $B(0, \sigma) \supseteq B(0, r_\infty)$ with $C = T_\infty$ if (l_∞) holds.

Indeed, since $T_\alpha\,|_K$ is k-set-contractive with $k < 1$, $(I - T_\alpha)|_K$ is closed on closed bounded sets for $\alpha = 0$ or $\alpha = \infty$. Hence $(I - T_\alpha)(\partial_K B(0, 1))$ is closed and thus, since $0 \notin (I - T_\alpha)(\partial_K B(0, 1))$, there exists $m_\alpha > 0$ such that $\| x - T_\alpha x \| \geqslant m_\alpha \| x \|$ for all x in K. Choose $r_\alpha > 0$ with $r_0 < r$ and $r_\infty > r$ such that $\| Tx - T_0(x)\| \leqslant (m_0/2) \| x \|$ for $x \in \bar{B}_K(0, r_0)$ if (l_0) holds and $\| Tx - T_\infty(x)\| < (m_\infty/2) \| x \|$ for $\| x \| \geqslant r_\infty$ if (l_∞) holds. Consequently for each $\lambda \in [0, 1]$ and any fixed $\sigma \in (0, r_0)$ if (l_0) holds or $\sigma > r_\infty$ if (l_∞) holds, we see that the map $\lambda T + (1 - \lambda) T_\alpha$ has no fixed points on $\partial_K B(0, \sigma)$. Indeed, in either case, for each $x \in \partial_K B(0, \sigma)$ and $\lambda \in [0, 1]$ we have the inequality

$$\| x - \lambda Tx - (1 - \lambda) T_\alpha x \| \geqslant \| x - T_\alpha x \| - \| Tx - T_\alpha x \| \geqslant \frac{m}{2} \sigma > 0.$$

In view of this and our hypothesis that T_α has no eigenvectors in $K \backslash \{0\}$ corresponding to eigenvectors greater than 1, we see that (i) of Corollary 1 follows from either (l_0) with $C = T_0$ and $D^1 = B(0, \sigma) \subsetneqq B(0, r_0)$ or from (l_∞) with $C = T_\infty$ and $D^1 = B(0, \sigma) \supset B(0, r_\infty)$. Q.E.D.

Remark 4. If in Corollaries 2 and 3 we are concerned only with the conditions (jj_0) and (l_0) respectively, then it suffices to assume in this case that T is defined only on $\bar{B}_K(0, r)$.

Corollaries 2 and 3 extend a number of results obtained earlier by other authors. We shall mention here some of them.

(1) Thus, if in condition (i) of Corollary 2 we set $C(x) \equiv 0$ and in conditon (ll) of Corollary 3 we set $F(x) \equiv w$ for some w in $K \backslash \{0\}$, we deduce Amann's recent Theorem 13.2 in [1] involving conditions (jj_0) and (l_0) (and its counterpart indicated in [1] involving conditions (jj_∞) and (l_∞)) stated explicitly in [1] for T compact. We add that our Corollary 2 also extends [9, Theorem 1.6] where T is compact and satisfies (j_0) and (jj_0).

(2) Another consequence of Corollaries 2 and 3 is the following result stated here for the case when $T(0) = 0$, which was first proved in [2, Theorem 2].

Corollary 4. *Suppose T: $K \to K$ is k-set-contractive with $k \in [0, 1)$ and asymptotically linear along K. Suppose also that $T(0) = 0$ and T is differentiable at 0 along K. Then T has a fixed point in $K \backslash \{0\}$ provided one of the following conditions holds*:

(a) $T_0 \in L_1^-(X)$ *and* $T_\infty \in L_1^+(X)$

(b) $T_0 \in L_1^+(X)$ *and* $T_\infty \in L_1^-(X)$.

Proof. (a) Let $T_0 \in L_1^-(X)$. Then, as was shown in the proof of Corollary 3, there exists $r_0 > 0$ such that $T_0 : B_K(0, r_0) \to K$ is k-set-contractive and $x \neq \lambda Tx + (1 - \lambda) T_0 x$ if $x \in \partial_K B(0, r_0)$ and $\lambda \in [0, 1]$. Since, by our hypothesis, $T_0 x \neq \mu x$ for $\mu \geqslant 1$ and $x \in K \backslash \{0\}$, it follows that (j) of Corollary 2 holds with

$C = T_0$ and $k_1 = k$. But $T_\infty \in L_1^+(X)$, i.e., (jj_∞) of Corollary 2 holds. Hence, when (a) holds, Corollary 4 follows from Corollary 2.

(b) Now, let $T_0 \in L_1^+(X)$. Then, as was shown in the proof of Corollary 2, there exist $r_0 > 0$ and $w \in K\backslash\{0\}$ such that F: $\bar{B}_K(0, r_0) \to K$ defined by $F(x) \equiv w$ for $x \in \bar{B}_K(0, r_0)$ satisfies (ll) of Corollary 3. Since $T_\infty \in L_1^-(X)$, by hypothesis, (l_∞) of Corollary 3 holds and thus, when (b) holds, Corollary 4 follows from Corollary 3. Q.E.D.

Finally, the results in [11, Theorems 4.11 and 4.16] and in [5, Theorem 6] also follow from our Corollary 1 since they are included in Corollary 4 as was noted in [2].

Finally let us remark that the results of Section 2 remain valid if instead of the set-measure γ we use the ball-measure of noncompactness χ to define k-ball-contractive mappings appearing in Corollaries 2, 3 and 4. If T: $K \to X$ is k-ball-contractive and asymptotically linear along K, then the proof that $T_\infty|_K$ is also k-ball-contractive is, with obvious modifications, the same as the proof of Proposition 1 in [15] or Lemma 1 in [2].

References

1. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18** (1976), 620–709.
2. H. Amann, Fixed points of asymptotically linear maps in ordered Banach spaces, *J. Functional Analysis* **14** (1973), 162–171.
3. J. Daneš, On densifying and related maps and their application in nonlinear functional analysis, "Theory of Nonlinear Operators, Proceedings Summer School" pp. 15–56, Akademie–Verlag, Berlin, 1974.
4. G. Darbo, Punti uniti in transformazioni a condominio non compatto, *Rend. Sem. Mat. Univ. Padova* **24** (1955), 84–92.
5. D. E. Edmunds, A. J. B. Potter, and C. A. Stuart, Noncompact positive operators, *Proc. Roy. Soc. London Ser. A* **328** (1972), 67–81.
6. P. M. Fitzpatrick and W. V. Petryshyn, Fixed point theorems and the fixed point index for multivalued mappings in cones, *J. London Math. Soc.* **12** (1975), 75–85.
7. P. M. Fitzpatrick and W. V. Petryshyn, Positive eigenvalues for nonlinear multivalued operators with applications to differential operators, *J. Differential Equations* **22** (1976), 428–441.
8. M. Furi and A. Vignoli, A fixed point theorem in complete metric spaces, *Bull. Un. Mat. Ital. Ser. IV* No 4–5 (1969), 505–506.
9. J. A. Gatica and A. L. Smith, Fixed point techniques in a cone with applications, *J. Math. Anal. Appl.* **61** (1977), 67–81.
10. G. B. Gustafson and I. Schmitt, Nonzero solutions of boundary value problems for second order ordinary and delay-differential equations, *J. Differential Equations* **12** (1972), 129–147.
11. M. A. Krasnoselskii, "Positive Solutions of Operator Equations," Noordhoff, Groningen, 1964.
12. P. S. Milojevič, Multivalued A-proper mappings and fixed point theorems for multivalued P_γ-compact mappings, to appear.

13. R. N. Nussbaum, The fixed point index for local condensing maps, *Ann. Mat. Pura Appl.* **89** (1971), 217–258.
14. R. N. Nussbaum, Periodic solutions of some nonlinear autonomous functional differential equations, *J. Differential Equations* **14** (1973), 360–394.
15. W. V. Petryshyn, Fredholm alternative for nonlinear k-ball contractive mappings with applications, *J. Differential Equations* **17** (1975), 82–95.
16. A. J. B. Potter, A fixed point theorem for positive k-set-contractions, *Proc. Edinburgh Math. Soc.* **19** (1974), 93–102.
17. B. N. Sadovskii, Limit compact and condensing operators, *Uspehi Mat. Nauk* **27** (1972), 81–146.
18. R. E. L. Turner, Transversality and cone maps, *Arch. Rational Mech. Anal.* **58** (1975), 151–179.